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Extremal Graphs with Prescribed Covering Number*

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We show that any connected graph with ϵ edges and covering number β satisfies the inequality $\epsilon \geqslant 2\beta - 1$. The graphs for which equality holds are characterized.

We use the notation and terminology of [2]. In particular, if G is a graph, $V(G)$ and $E(G)$ denote its vertex and edge sets, and $\nu(G)$ and $\epsilon(G)$ the cardinalities of these sets. A *covering* of G is a set $K \subseteq V$ such that each edge of G has at least one end in K. The minimum number of vertices in a covering of G is denoted by $\beta(G)$ and is called the *covering number* of G.

Erdös, Hajnal, and Moon [3] proved that, for any simple graph G, $\epsilon \leqslant \binom{\beta+1}{2}$, and that the unique graph G with $\epsilon = \binom{\beta+1}{2}$ is $K_{\beta+1}$ (apart from isolated vertices). In this paper, we look at the minimum number of edges in a graph with prescribed covering number.

It is easily seen that every graph G satisfies $\epsilon \geqslant \beta$ and that $\epsilon = \beta$ if and only if (apart from isolated vertices) G consists of β copies of K_2.

THEOREM. *If G is connected, then $\epsilon \geqslant 2\beta - 1$. Moreover, $\epsilon = 2\beta - 1$ if and only if, for some nonnegative integers l and m, G consists of l K_2 components and m odd cycle components linked together by $l + m - 1$ edges.*

To prove the theorem, we need a lemma, a proof of which can be found in [1, p.287]. A graph G is *β-critical* if $\beta(G - e) < \beta(G)$ for all $e \in E(G)$. (Since $\alpha + \beta = \nu$, such graphs are identical to the α-critical graphs defined in [1].)

LEMMA. *Every connected β-critical graph is a block.*

Proof of theorem. We use induction on ν to show that $\epsilon \geqslant 2\beta - 1$. This is clearly so if $\nu = 1$ or $\nu = 2$. Suppose it is true for all graphs with fewer than n vertices, and let $\nu(G) = n > 2$. Let $P = v_1 v_2 \cdots v_m$ be a path of maximum length in G. Clearly, $d_G(v_2) \geqslant 2$. We show that $G - v_2$ is con-

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nected, apart from possible isolated vertices. In $G - v_2$, the vertices $v_3, v_4, \ldots, v_m$ clearly belong to one component C_1. Suppose there were some other component C_2 containing at least one edge. In C_2 there would be a vertex w_1 joined (in G) to v_2, and an edge w_0w_1. However, this would imply the existence in G of the path $w_0w_1v_2 \cdots v_m$, which would be longer than P. Hence $G - v_2$ is connected (apart from possible isolated vertices). By the induction hypothesis, $\epsilon(G - v_2) \geqslant 2\beta(G - v_2) - 1$. Hence

$$\epsilon(G) = \epsilon(G - v_2) + d_G(v_2) \geqslant 2\beta(G - v_2) + 1 \geqslant 2\beta(G) - 1.$$

We prove the second statement by induction on the number of cut edges of G. Suppose that $\epsilon = 2\beta - 1$. If G is β-critical, then, by the lemma, G is a block. Thus either $G \cong K_2$ or G is 2-connected. In the latter case, let v be any vertex of G. Since $G - v$ is connected, $\epsilon(G - v) \geqslant 2\beta(G - v) - 1$. Because $\beta(G - v) \geqslant \beta(G) - 1$, it follows that the degree of v is at most 2, and hence exactly 2. Therefore, in this case, G must be a cycle of length $2\beta - 1$.

If G is not β-critical, let e be an edge such that $\beta(G - e) = \beta(G)$, and set $H = G - e$. Clearly,

$$\epsilon(H) = 2\beta(H) - 2 \tag{1}$$

and so e must be a cut edge of G. Let H_1 and H_2 be the components of H. Then

$$\epsilon(H_1) \geqslant 2\beta(H_1) - 1 \tag{2}$$

and

$$\epsilon(H_2) \geqslant 2\beta(H_2) - 1 \tag{3}$$

But, since $\epsilon(H) = \epsilon(H_1) + \epsilon(H_2)$ and $\beta(H) = \beta(H_1) + \beta(H_2)$, (1) forces equality in (2) and (3). On applying the induction hypothesis to H_1 and H_2, we see that G has the stated description. It is easy to show that, conversely, every graph G of this form satisfies $\epsilon = 2\beta - 1$.

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